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# TRANSIENT, ASYMPTOTIC, ELASTODYNAMIC ANALYSIS: A SIMPLE METHOD AND ITS APPLICATION TO MIXED-MODE CRACK GROWTH

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**Abstract**—When a moving singularity in a linearly elastic solid admits, in its steady state, an asymptotic eigen-expansion of variable-separable type, its asymptotic expansion in an unsteady state can be obtained systematically. A simple, straightforward method is presented here for deriving the transient, asymptotic, elastodynamic local fields near such moving singularities. The method makes use of the steady-state solution, and turns a transient problem into finding the particular solutions of a set of second-order ordinary differential equations, which have constant coefficients and nonhomogeneous terms involving only cosine and sine functions. The method is employed to obtain transient plane elastodynamic near-tip fields for a crack growing in a homogeneous, isotropic and linearly elastic solid. The fields are given in terms of two displacement potentials, applicable to general mixed-mode crack growth cases, from which crack-tip stress and deformation fields can be readily derived. Such transient solutions will be useful in numerical simulation as well as experimental interpretation of dynamic crack propagation events.

## INTRODUCTION

The transient feature of mode I plane elastodynamic near-tip fields during crack propagation has recently been investigated by Freund and Rosakis (1992) and Rosakis *et al.* (1991). It is found that the square-root-singular term alone in the *steady-state* asymptotic expansion of the crack-tip fields does not fully describe the crack-tip state during *transient* crack growth, which is attributed to the dependence of local fields on the past history of time-dependent quantities such as crack velocity and stress intensity factor. They demonstrated experimentally that by incorporating transient higher-order expansions an accurate description of crack-tip fields under fairly severe transient conditions can be achieved.

The availability of series expansions of transient crack-tip fields will provide a strong foundation for the interpretation of experimental measurements in dynamic fracture testing. The importance of utilizing multi-term field expansions, such as that for steady-state crack growth (Nishioka and Atluri, 1983), can be seen from the pioneering work of Sanford and Dally (1979), as well as a more recent study by Chao *et al.* (1992), among others. Inclusion of transient terms in the field expansions will naturally improve the accuracy and reliability of the aforementioned experimental investigations. Apparently, the usefulness and necessity of transient higher-order asymptotic expansions are certainly not confined to propagating cracks or to the interpretation of experimental measurements. They can be extended to describe responses of stationary cracks and other singularities under dynamic loading conditions, and can be used effectively in numerical simulations of such singularities during transient dynamic events. Further effort in this area is well-justified and worth pursuing.

In this article, a general and systematic method, which provides an alternative to that used in Freund and Rosakis (1992), is proposed for the derivation of transient, asymptotic, elastodynamic local fields near moving singularities such as growing crack tips. In this method, the transient solution is treated as the sum of that of the counterpart steady-state problem, modified slightly, and a correction term that is solved as a particular solution of an ordinary differential equation. The method is used to obtain transient near-tip fields, in terms of two displacement potentials, for a crack propagating in a homogeneous, isotropic and linearly elastic solid under general plane (plane strain or plane stress) deformation conditions. Explicit expressions for displacements and stresses in crack-tip polar coordinates can be readily derived from these two displacement potentials obtained in this study. These mixed-mode results are complementary to the mode I solutions of Freund and Rosakis. Counterpart transient analyses for a stationary crack under dynamic loading, for all three fracture modes, are more involved mathematically and are reported separately (Deng, 1992a).

## PROBLEM FORMULATION

Consider a plane elastodynamic problem in a domain occupied by a homogeneous, isotropic, and linearly elastic solid. In a rectangular $(x, y)$ coordinate system, displacements $u_x$ and $u_y$, and stresses $\sigma_x$, $\sigma_y$ and $\sigma_{xy}$, can be expressed as [see, for example, Freund (1990)]

$$u_x = \frac{\partial\phi}{\partial x} + \frac{\partial\psi}{\partial y},$$

$$u_y = \frac{\partial\phi}{\partial y} - \frac{\partial\psi}{\partial x}, \tag{1}$$

$$\frac{\sigma_x}{\mu} = \frac{k+1}{k-1}\frac{\partial^2\phi}{\partial x^2} + \frac{3-k}{k-1}\frac{\partial^2\phi}{\partial y^2} + 2\frac{\partial^2\psi}{\partial x\,\partial y},$$

$$\frac{\sigma_y}{\mu} = \frac{3-k}{k-1}\frac{\partial^2\phi}{\partial x^2} + \frac{k+1}{k-1}\frac{\partial^2\phi}{\partial y^2} - 2\frac{\partial^2\psi}{\partial x\,\partial y},$$

$$\frac{\sigma_{xy}}{\mu} = -\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + 2\frac{\partial^2\phi}{\partial x\,\partial y}, \tag{2}$$

where $\mu$ is the shear modulus; $k = (3-v)/(1+v)$ in plane stress and $(3-4v)$ in plane strain, $v$ being the Poisson's ratio; $\phi$ and $\psi$, both functions of position $(x, y)$ and time $t$, are displacement potentials satisfying the following wave equations:

$$(D_l + D_t)\phi = 0, \quad (D_s + D_t)\psi = 0, \tag{3}$$

where $D_l$, $D_s$ and $D_t$ are differential operators. For the sake of asymptotic analysis, we assume that a point singularity is moving along a straight line, say, the positive $x$-axis, with speed $v(t)$ at time $t$. Now let the coordinate system sit at and move with the singularity relative to an inertia frame. Then the differential operators $D_l$, $D_s$ and $D_t$ can be written as:

$$D_\beta = c_\beta^2\left(\alpha_\beta^2\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \quad (\beta = l, s), \qquad D_t = 2\sqrt{v}\frac{\partial}{\partial t}\left(\sqrt{v}\frac{\partial}{\partial x}\right) - \frac{\partial^2}{\partial t^2}, \tag{4}$$

where $c_s = (\mu/\rho)^{1/2}$, $c_l = [(k+1)/(k-1)]^{1/2}c_s$ and $\alpha_\beta = [1-(v/c_\beta)^2]^{1/2}$, $\beta = l, s$. If we define the *distorted* polar coordinates $(r_\beta, \theta_\beta)$ as those associated with the *scaled* rectangular coordinates $(x, \alpha_\beta y)$, where $\beta = l$ or $s$, then we can write the differential operators for $\phi$ as

$$D_l = \alpha_l^2 c_l^2\left(\frac{\partial^2}{\partial r_l^2} + \frac{1}{r_l}\frac{\partial}{\partial r_l} + \frac{1}{r_l^2}\frac{\partial^2}{\partial\theta_l^2}\right),$$

$$D_t = 2\sqrt{v}\frac{\partial}{\partial t}\left[\sqrt{v}\left(\cos\theta_l\frac{\partial}{\partial r_l} - \frac{\sin\theta_l}{r_l}\frac{\partial}{\partial\theta_l}\right)\right] - \frac{\partial^2}{\partial t^2}. \tag{5}$$

Those for $\psi$ are obtained by switching subscript $l$ to $s$ in eqn (5). As cautioned in Freund and Rosakis (1992), $\alpha_\beta$, $r_\beta$ and $\theta_\beta$ $(\beta = l, s)$ will depend on time $t$ if speed $v$ is not constant.

When the time derivatives $D_t\phi$ and $D_t\psi$ are identically zero, equations in (3) become harmonic and their solutions can be expressed in terms of analytic functions. These solutions are said to be *steady-state* for a moving singularity. When the time derivatives in (3) are not negligible, solutions obtained are called *transient*.

## SOLUTION METHOD

Consider a point singularity such as the tip of a moving wedge or crack, and assume that it admits a steady-state asymptotic solution of variable-separable type. Such solutions are readily obtainable with well-established techniques such as eigen-expansion (Williams, 1952, 1957) and complex-variable representation (Muskhelishvili, 1953; Radok, 1956). For example, complete near-tip expansions have been given by Nishioka and Atluri (1983) for steadily propagating cracks in homogeneous, isotropic, and linearly elastic solids; and by Deng (1991, 1992b) for steadily-propagating interfacial cracks in dissimilar isotropic as well as anisotropic elastic bimaterials. The steady-state solutions will be denoted here as $\phi_0$ and $\psi_0$, which, if admitting variable-separable eigen-expansions, will have the following form:

$$\phi_0 = \sum_{n=1}^{\infty} [a_{n1} \cos \lambda_n \theta_l + a_{n2} \sin \lambda_n \theta_l] r_l^{\lambda_n},$$

$$\psi_0 = \sum_{n=1}^{\infty} [b_{n1} \cos \lambda_n \theta_s + b_{n2} \sin \lambda_n \theta_s] r_s^{\lambda_n}, \tag{6}$$

where $\lambda_n$ $(n = 1, 2, \ldots)$ are eigenvalues, with real parts arranged according to $\mathrm{Re}(\lambda_1) < \mathrm{Re}(\lambda_2) < \cdots$, and coefficients $a_{n1}$, $a_{n2}$, $b_{n1}$ and $b_{n2}$ are related such that local traction and displacement boundary conditions at the singularity are satisfied.

Under transient conditions, the steady-state solution must be modified to include any disturbances due to transient effects. A first-order approximation would be to let coefficients $a_{n1}$, $a_{n2}$, $b_{n1}$ and $b_{n2}$ in eqn (6) be time dependent. Since both $\phi$ and $\psi$ are governed by the same type of equations, the following method, which equally applies to $\phi$ and $\psi$, will be discussed for $\phi$ only. Now to derive the transient solution from the steady-state one, let

$$\phi(r_l, \theta_l, t) = \phi_0(r_l, \theta_l, t) + \phi^*(r_l, \theta_l, t), \tag{7}$$

where $\phi^*$ is a correction term and $\phi_0$ now has time-dependent coefficients. Substitution of eqn (7) into (3) yields this *nonhomogeneous* governing equation for $\phi^*$

$$(D_l + D_t)\phi^* = -D_t \phi_0. \tag{8}$$

The *advantage* of this method is that the form of the solution for $\phi^*$ is indicated by that of $\phi_0$, which is available in eqn (6). It can be seen that substitution of eqn (6) into (8) will yield a nonzero right-hand side with a variable-separable expansion in $r_l$ and $\theta_l$. This feature, along with the polar form of operator $D_l$ as shown in (5), suggests that a solution for $\phi^*$ will also admit a similar expansion, whose exact form depends on the relations between the eigenvalues. For example, for cracks and 180°-angled wedges under various local crack-surface or wedge-edge boundary conditions, the following expansion can be made:

$$\phi^* = \sum_{n=1}^{\infty} f_n(\theta_l, t) r_l^{\lambda_n}. \tag{9}$$

From eqns (6), (8) and (9), and by collecting terms with the same powers of $r_l$, a set of *linear ordinary differential equations* can be obtained for $f_n(\theta_l, t)$, *with time t as a parameter*. These equations are homogeneous for the first few (two or four) leading terms, and nonhomogeneous for all other higher order terms. The solutions of these equations are composed of a homogeneous part and a particular part, which are zero for the first few leading terms. It can be shown that the forms of the homogeneous solutions of these equations are identical to those in eqn (6), meaning that they can be incorporated into (6) by modifying the coefficients $a_{n1}(t)$, $a_{n2}(t)$, $b_{n1}(t)$ and $b_{n2}(t)$. The relations between these coefficients *must be updated* from local boundary conditions with the effects of the transient term $\phi^*$ taken into consideration. *Therefore, to find $\phi$ one only needs to modify the expansion coefficients in $\phi_0$, and find, for $\phi^*$, the particular solutions of a set of nonhomogeneous, linear, ordinary differential equations, usually only involving cosines and sines of $\theta_l$.* This method is now used to obtain the transient elastodynamic near-tip fields for a moving crack.

## PROPAGATING CRACK-TIP FIELDS

Consider a crack that is propagating at a speed $v(t)$ along the $x$-axis in a homogeneous, isotropic, and linearly elastic solid and under general mixed-mode conditions. Suppose the coordinate system is situated at, and translates with the crack tip, with the negative $x$-axis coinciding with the traction-free crack surfaces. The eigenvalues given by Williams (1957) for stationary cracks are still valid in this case [for example, see Nishioka and Atluri (1983)], that is, $\lambda_n = n/2+1$ $(n = 1, 2, \ldots)$. Using the proposed method, one has

$$\phi_0 = \sum_{n=1}^{\infty} [a_{n1}\cos(n/2+1)\theta_l + a_{n2}\sin(n/2+1)\theta_l] r_l^{n/2+1},$$

$$\phi^* = \sum_{n=1}^{\infty} f_n(\theta_l, t) r_l^{n/2+1}, \tag{10}$$

where it is noted again that $r_l$ and $\theta_l$ will depend on time $t$ if the crack speed $v$ is not constant. To account for this time dependence, note the following relations:

$$\frac{\mathrm{d}\alpha_l}{\mathrm{d}t} = -\frac{v}{\alpha_l c_l^2}\frac{\mathrm{d}v}{\mathrm{d}t}, \quad \frac{\partial\theta_l}{\partial t} = -\frac{v}{2\alpha_l^2 c_l^2}\frac{\mathrm{d}v}{\mathrm{d}t}\sin 2\theta_l,$$

$$\frac{\partial r_l}{\partial t} = -\frac{v}{2\alpha_l^2 c_l^2}\frac{\mathrm{d}v}{\mathrm{d}t}(1-\cos 2\theta_l) r_l. \tag{11}$$

Substituting (10) into (8), with the use of (5) and (11), one obtains the following set of nonhomogeneous, linear, ordinary differential equations for the particular solutions of $f_n(\theta_l, t)$:

$$L_n f_n = 0, \quad (n = 1, 2), \quad L_n = \frac{\partial^2}{\partial\theta_l^2} + (n/2+1)^2, \tag{12}$$

$$\begin{aligned}
-\alpha_l^2 c_l^2 L_n f_n = {} & \left[\frac{n\dot{v}}{2}\left(1-\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\right)a_{(n-2)1} + nv\dot{a}_{(n-2)1}\right]\cos(n/2-1)\theta_l \\
& + \left[\frac{n\dot{v}}{2}\left(1-\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\right)a_{(n-2)2} + nv\dot{a}_{(n-2)2}\right]\sin(n/2-1)\theta_l \\
& + \frac{n\dot{v}}{2}\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}[a_{(n-2)1}\cos(n/2-3)\theta_l + a_{(n-2)2}\sin(n/2-3)\theta_l], \quad (n = 3, 4),
\end{aligned} \tag{13}$$

$$\begin{aligned}
-\alpha_l^2 c_l^2 L_n f_n = {} & -\ddot{f}_{n-4} + nv\cos\theta_l \dot{f}_{n-2} - 2v\sin\theta_l \frac{\partial \dot{f}_{n-2}}{\partial\theta_l} - (\ddot{a}_{(n-4)1} - nv\dot{a}_{(n-2)1})\cos(n/2-1)\theta_l \\
& - (\ddot{a}_{(n-4)2} - nv\dot{a}_{(n-2)2})\sin(n/2-1)\theta_l + T_n, \quad (n \geqslant 5),
\end{aligned} \tag{14}$$

where $T_n$ is a collection of terms related to $f_{n-2}$, $f_{n-4}$, $a_{(n-2)1}$, $a_{(n-2)2}$, $a_{(n-4)1}$, $a_{(n-4)2}$, $v$, their first and/or second order derivatives, and cosine and sine functions of $\theta_l$, and it vanishes when the crack speed $v$ is constant. The expression for $T_n$ is lengthy and is listed in the Appendix. Note that *one* superimposed *dot* implies differentiation *once* with respect to time, and that the dependence of $r_l$ and $\theta_l$ on time $t$ has been accounted for in the above equations; hence $r_l$, $\theta_l$ and $t$ must be treated as independent variables. It is observed that $L_n$ is a simple second-order ordinary differential operator, and it can be seen that the nonhomogeneous terms on the right-hand side of the equations are only sine and cosine functions of $\theta$. This makes finding the particular solutions of the above equations very easy since, for an arbitrary value $\xi$ not equal to $(n/2+1)$, the following holds:

$$L_n^{-1}\cos\xi\theta_l = \cos\xi\theta_l/[(n/2+1)^2-\xi^2], \quad L_n^{-1}\sin\xi\theta_l = \sin\xi\theta_l/[(n/2+1)^2-\xi^2]. \quad (15)$$

It is clear that the particular solutions for the first two leading terms $f_n$ $(n = 1,2)$ are zero, and those for $n = 3,4$ are found to be

$$f_n = -\frac{1}{4\alpha_l^2 c_l^2}\left\{\left[2v\dot{a}_{(n-2)1}+\dot{v}\left(1-\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\right)a_{(n-2)1}\right]\cos\left(\frac{n}{2}-1\right)\theta_l\right.$$
$$+\left[2v\dot{a}_{(n-2)2}+\dot{v}\left(1-\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\right)a_{(n-2)2}\right]\sin\left(\frac{n}{2}-1\right)\theta_l$$
$$\left.+\frac{nv^2\dot{v}}{4\alpha_l^2 c_l^2}\left[a_{(n-2)1}\cos\left(\frac{n}{2}-3\right)\theta_l+a_{(n-2)2}\sin\left(\frac{n}{2}-3\right)\theta_l\right]\right\}. \quad (16)$$

In general, solutions for $f_n$ and $f_{n+1}$, where $n$ is an odd number and $n \geqslant 5$, can be easily obtained in terms of a common expression by solving eqn (14) progressively from those $f_n$ pairs that are already known. For example, for the simpler case of transient crack growth with constant crack speed (thus $T_n = 0$), solutions for $f_n$ $(n = 1\text{–}4)$ can be substituted in (14) to yield the following expression for $f_n$ $(n = 5,6)$:

$$f_n = -\frac{1}{2n\alpha_l^2 c_l^2}\left\{\left[nv\dot{a}_{(n-2)1}-\left(1+\frac{v^2}{\alpha_l^2 c_l^2}\right)\ddot{a}_{(n-4)1}\right]\cos(n/2-1)\theta_l\right.$$
$$+\left[nv\dot{a}_{(n-2)2}-\left(1+\frac{v^2}{\alpha_l^2 c_l^2}\right)\ddot{a}_{(n-4)2}\right]\sin(n/2-1)\theta_l$$
$$\left.-\frac{nv^2}{4\alpha_l^2 c_l^2}\ddot{a}_{(n-4)1}\cos(n/2-3)\theta_l-\frac{nv^2}{4\alpha_l^2 c_l^2}\ddot{a}_{(n-4)2}\sin(n/2-3)\theta_l\right\}. \quad (17)$$

Equivalent expressions for $f_n$ have been given earlier by Freund and Rosakis (1992) under mode I plane stress conditions for $n = 1,2$ and 3 when crack speed is not constant and for $n = 1\text{–}6$ when crack speed is constant.

It is noted that solutions for the other displacement potential $\psi$ can be obtained from those for $\phi$ by changing subscript $l$ to $s$, functions $f_n$ to $g_n$, and coefficients $a_{n1}$ and $a_{n2}$ to $b_{n1}$ and $b_{n2}$. It is also noted that the out-of-plane displacement $u_z$ for mode III crack propagation is governed by the same wave equation as that for $\psi$, and hence will have the same form of transient asymptotic expansion, except that $n$ starts from $-1$. With solutions for the displacement potentials available, the transient crack-tip asymptotic expansions for stresses and displacements can be readily derived by substituting the expressions for the potentials into eqns (1) and (2) and by noting

$$\frac{\partial}{\partial x} = \cos\theta_\beta\frac{\partial}{\partial r_\beta}-\frac{\sin\theta_\beta}{r_\beta}\frac{\partial}{\partial\theta_\beta},$$
$$\frac{\partial}{\partial y} = \alpha_\beta\left(\sin\theta_\beta\frac{\partial}{\partial r_\beta}+\frac{\cos\theta_\beta}{r_\beta}\frac{\partial}{\partial\theta_\beta}\right), \quad (\beta = l,s). \quad (18)$$

Coefficients $a_{n1}$, $a_{n2}$, $b_{n1}$ and $b_{n2}$ are related through the traction-free conditions along the crack surfaces, and it can be shown that those for $n = 1$ are related to the instantaneous dynamic stress intensity factors $K_{\rm I}(t)$ and $K_{\rm II}(t)$. For brevity, explicit expressions for the mixed-mode stress and displacement fields are not attempted here. However, a three-term ($n = 3$) expansion of the stress field for nonuniform mode I crack growth can be found in Rosakis *et al.* (1991).

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## APPENDIX: EXPRESSION FOR $T_n$ OF EQN (14)

$$T_n = \eta_1 f_{n-2} + \eta_2 \frac{\partial f_{n-2}}{\partial \theta_l} + \eta_3 \frac{\partial^2 f_{n-2}}{\partial \theta_l^2} + \eta_4 f_{n-4} + \eta_5 \bar{f}_{n-4} + \eta_6 \frac{\partial f_{n-4}}{\partial \theta_l} + \eta_7 \frac{\partial \bar{f}_{n-4}}{\partial \theta_l}$$
$$+ \eta_8 \frac{\partial^2 f_{n-4}}{\partial \theta_l^2} + \eta_9 \cos(n/2-1)\theta_l + \eta_{10} \sin(n/2-1)\theta_l + \eta_{11} \cos(n/2-3)\theta_l$$
$$+ \eta_{12} \sin(n/2-3)\theta_l + \eta_{13} \cos(n/2-5)\theta_l + \eta_{14} \sin(n/2-5)\theta_l,$$

where $\eta_k$ ($k = 1$–$14$) are given by

$$\eta_1 = \frac{n\dot{v}}{2}\left[\left(1 - \frac{(n-4)v^2}{4\alpha_l^2 c_l^2}\right)\cos\theta_l + \frac{(n-4)v^2}{4\alpha_l^2 c_l^2}\cos 3\theta_l\right],$$

$$\eta_2 = -\dot{v}\left[\left(1 - \frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\right)\sin\theta_l + \frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\sin 3\theta_l\right],$$

$$\eta_3 = \frac{\dot{v}v^2}{2\alpha_l^2 c_l^2}(\cos\theta_l - \cos 3\theta_l),$$

$$\eta_4 = \frac{(n-2)}{32\alpha_l^4 c_l^4}[8\alpha_l^2 c_l^2(v\ddot{v}+\dot{v}^2) - 3(n-6)\dot{v}^2 v^2] - \frac{(n-2)}{8\alpha_l^4 c_l^4}[2\alpha_l^2 c_l^2(v\ddot{v}+\dot{v}^2) - (n-6)\dot{v}^2 v^2]\cos 2\theta_l$$
$$- \frac{(n-2)(n-6)\dot{v}^2 v^2}{32\alpha_l^4 c_l^4}\cos 4\theta_l,$$

$$\eta_5 = \frac{(n-2)\dot{v}v}{2\alpha_l^2 c_l^2}(1 - \cos 2\theta_l),$$

$$\eta_6 = \frac{1}{4\alpha_l^4 c_l^4}[2\alpha_l^2 c_l^2(v\ddot{v}+\dot{v}^2) - (n-6)\dot{v}^2 v^2]\sin 2\theta_l + \frac{(n-4)\dot{v}^2 v^2}{8\alpha_l^4 c_l^4}\sin 4\theta_l,$$

$$\eta_7 = \frac{\dot{v}v}{\alpha_l^2 c_l^2}\sin 2\theta_l,$$

$$\eta_8 = -\frac{\dot{v}v^2}{8\alpha_l^4 c_l^4}(1-\cos 4\theta_l),$$

$$\eta_9 = \frac{n\dot{v}}{2}\left(1-\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\right)a_{(n-2)1} + \frac{(n-2)\dot{v}v}{2\alpha_l^2 c_l^2}\dot{a}_{(n-4)1},$$

$$\eta_{10} = \frac{n\dot{v}}{2}\left(1-\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}\right)a_{(n-2)2} + \frac{(n-2)\dot{v}v}{2\alpha_l^2 c_l^2}\dot{a}_{(n-4)2},$$

$$\eta_{11} = \frac{n\dot{v}}{2}\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}a_{(n-2)1} - \frac{(n-2)\dot{v}v}{2\alpha_l^2 c_l^2}\dot{a}_{(n-4)1} - \frac{(n-2)}{8\alpha_l^4 c_l^4}[2\alpha_l^2 c_l^2(v\ddot{v}+\dot{v}^2)-(n-6)\dot{v}^2 v^2]a_{(n-4)1},$$

$$\eta_{12} = \frac{n\dot{v}}{2}\frac{(n-2)v^2}{2\alpha_l^2 c_l^2}a_{(n-2)2} - \frac{(n-2)\dot{v}v}{2\alpha_l^2 c_l^2}\dot{a}_{(n-4)2} - \frac{(n-2)}{8\alpha_l^4 c_l^4}[2\alpha_l^2 c_l^2(v\ddot{v}+\dot{v}^2)-(n-6)\dot{v}^2 v^2]a_{(n-4)2},$$

$$\eta_{13} = -\frac{(n-2)(n-4)\dot{v}^2 v^2}{16\alpha_l^4 c_l^4}a_{(n-4)1},$$

$$\eta_{14} = -\frac{(n-2)(n-4)\dot{v}^2 v^2}{16\alpha_l^4 c_l^4}a_{(n-4)2}.$$

It is worth noting that the presence of the large number of terms in $T_n$ is solely due to the time dependence of the crack propagation speed $v$ and the coefficients, $a_{(n-2)1}$, $a_{(n-2)2}$, $a_{(n-4)1}$ and $a_{(n-4)2}$, for the original steady-state series expansions. As such, the transient field expansions do not introduce any more unknown coefficients than their steady-state counterparts, and all the extra terms, being associated with the time derivatives of the crack speed and existing coefficients, must be included, unless these time derivatives are identically zero, to account correctly for any transient effects during a nonuniform crack propagation event.